Unit 2
Creative formula manipulation

Creative formula manipulation is based in a large part on mastering the dialectic between freedom and determination in arithmetic:

- Freedom:
  * Commutative and associative laws of addition and multiplication are the basis for a free arrangement (and re-arrangement) and free structured bracketing of operands.
  * Free and creative use of the two directions of the identity laws and its derivatives

$$a+0=a$$
$$a-a=0$$
$$a*1=a$$
$$a/a=1$$
$$a*0=0$$
$$(-1)*(-1)=1$$

← ---------- →

extension ← insertion reduction

- Determination:
  * Distributive law of multiplication with respect to addition
  * Priority law + < * < ^ .

Example 1
Power sums and Pascal's triangle

Now is $\sum_{i=1}^{n} i^4 \rightarrow \sum_{i=1}^{n} i^m = p_m(n)$; $m, n \in \mathbb{N}$ on the stage.

We start light-footed with a simple identity

$4^k = 4^k$; $k \in \mathbb{N}_0$

Now we insert the additive identity 0 (invariant)

$4^k = 4^k + 0 + 0 + 0 - 0$.

Next step: extension of 0 by an opposite pair

$4^k = 4^k + (3^k-3^k) + (2^k-2^k) + (1^k-1^k) - 0^k$:

* "creatio ex nihilo"

* Pilgering process.

* A chess-player plans always some moves in advance.
Last step: reordering:

$$4^k = (4^k\text{-}3^k) + (3^k\text{-}2^k) + (2^k\text{-}1^k) + (1^k\text{-}0^k) = \sum_{i=1}^{4} i^k - (i\text{-}1)^k \ .$$

Now generalization $4 \rightarrow k$:

$n^k = \sum_{i=1}^{n} i^k - (i-1)^k \ ; \ n \in \mathbb{N} \, ; \, k \in \mathbb{N}_0$

This is our magic formula.
Using the binomial formula, we obtain

$$n^k = \sum_{i=1}^{n} [i^k - (i-1)^k] = \sum_{i=1}^{n} [i^k - \sum_{j=0}^{k-1} (\binom{k}{j} (-1)^{k-j} i^j) - i^k] =$$

$$= \sum_{i=1}^{n} [\sum_{j=0}^{k-1} (\binom{k}{j} (-1)^{k-1-j} i^j)] = \sum_{j=0}^{k-1} (\binom{k}{j} (-1)^{k-1-j} [\sum_{i=1}^{n} i^j])$$

and with

$$\sum_{i=1}^{n} i^j = f(j,n)$$

$$n^k = \sum_{j=0}^{k-1} \binom{k}{j} (-1)^{k-1-j} f(j,n) \ ; \ k=1,2,3,4,5,.. \ ,$$

detailed for k=1(1)5

$$+n^1 = \binom{1}{0} f(0,n)$$

$$-n^2 = \binom{2}{0} f(0,n) - \binom{2}{1} f(1,n)$$

$$+n^3 = \binom{3}{0} f(0,n) - \binom{3}{1} f(1,n) + \binom{3}{2} f(2,n)$$

$$-n^4 = \binom{4}{0} f(0,n) - \binom{4}{1} f(1,n) + \binom{4}{2} f(2,n) - \binom{4}{3} f(3,n)$$

$$+n^5 = \binom{5}{0} f(0,n) - \binom{5}{1} f(1,n) + \binom{5}{2} f(2,n) - \binom{5}{3} f(3,n) + \binom{5}{4} f(4,n) \ ,$$

in matrix representation

$$\begin{pmatrix} +n^1 \\ -n^2 \\ +n^3 \\ -n^4 \\ +n^5 \end{pmatrix} = \begin{pmatrix} +\binom{1}{0} & 0 & 0 & 0 & 0 \\ +\binom{2}{0} & -\binom{2}{1} & 0 & 0 & 0 \\ +\binom{3}{0} & -\binom{3}{1} & +\binom{3}{2} & 0 & 0 \\ +\binom{4}{0} & -\binom{4}{1} & +\binom{4}{2} & -\binom{4}{3} & 0 \\ +\binom{5}{0} & -\binom{5}{1} & +\binom{5}{2} & -\binom{5}{3} & +\binom{5}{4} \end{pmatrix} \begin{pmatrix} f(0,n) \\ f(1,n) \\ f(2,n) \\ f(3,n) \\ f(4,n) \end{pmatrix} .$$

Then f(4,n) can be found using Cramer's rule
http://mathworld.wolfram.com/CramersRule.html
after slight rearrangement as

$$f(4,n) = \sum_{i=1}^{n} i^4 = \frac{n}{(4+1)!} \begin{vmatrix} +n^0 & \binom{1}{0} & 0 & 0 & 0 \\ -n^1 & \binom{2}{0} & \binom{2}{1} & 0 & 0 \\ +n^2 & \binom{3}{0} & \binom{3}{1} & \binom{3}{2} & 0 \\ -n^3 & \binom{4}{0} & \binom{4}{1} & \binom{4}{2} & \binom{4}{3} \\ +n^{3+1} & \binom{4+1}{0} & \binom{4+1}{1} & \binom{4+1}{2} & \binom{4+1}{3} \end{vmatrix}$$

determinant of (4+1)th order

$$= \frac{n}{5!} \begin{vmatrix} +n^0 & 1 & 0 & 0 & 0 \\ -n^1 & 1 & 2 & 0 & 0 \\ +n^2 & 1 & 3 & 3 & 0 \\ -n^3 & 1 & 4 & 6 & 4 \\ +n^4 & 1 & 5 & 10 & 10 \end{vmatrix}$$

Pascal's triangle

determinant of 5th order

and after normalization

$$\frac{f(4,n)}{f(0,n)} = \frac{\sum_{i=1}^{n} i^4}{\sum_{i=1}^{n} i^0 = n} = \frac{\begin{vmatrix} +n^0 & \binom{1}{0} & 0 & 0 & 0 \\ -n^1 & \binom{2}{0} & \binom{2}{1} & 0 & 0 \\ +n^2 & \binom{3}{0} & \binom{3}{1} & \binom{3}{2} & 0 \\ -n^3 & \binom{4}{0} & \binom{4}{1} & \binom{4}{2} & \binom{4}{3} \\ +n^4 & \binom{4+1}{0} & \binom{4+1}{1} & \binom{4+1}{2} & \binom{4+1}{3} \end{vmatrix}}{\begin{vmatrix} +1 & \binom{1}{0} & 0 & 0 & 0 \\ -1 & \binom{2}{0} & \binom{2}{1} & 0 & 0 \\ +1 & \binom{3}{0} & \binom{3}{1} & \binom{3}{2} & 0 \\ -1 & \binom{4}{0} & \binom{4}{1} & \binom{4}{2} & \binom{4}{3} \\ +1 & \binom{4+1}{0} & \binom{4+1}{1} & \binom{4+1}{2} & \binom{4+1}{3} \end{vmatrix}} = (4+1)!$$

determinants of (4+1)th order

$$= \frac{1}{5!}\begin{vmatrix} +n^0 & 1 & 0 & 0 & 0 \\ -n^1 & 1 & 2 & 0 & 0 \\ +n^2 & 1 & 3 & 3 & 0 \\ -n^3 & 1 & 4 & 6 & 4 \\ +n^4 & 1 & 5 & 10 & 10 \end{vmatrix} = \frac{\begin{vmatrix} +n^0 & 1 & 0 & 0 & 0 \\ -n^1 & 1 & 2 & 0 & 0 \\ +n^2 & 1 & 3 & 3 & 0 \\ -n^3 & 1 & 4 & 6 & 4 \\ +n^4 & 1 & 5 & 10 & 10 \end{vmatrix}}{\begin{vmatrix} +1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 2 & 0 & 0 \\ +1 & 1 & 3 & 3 & 0 \\ -1 & 1 & 4 & 6 & 4 \\ +1 & 1 & 5 & 10 & 10 \end{vmatrix}} \qquad \text{(c)}$$

determinants of 5th order

$$= \sum_{j=0}^{4} A(4,j)\, n^j = p_4(n)$$

$$= \begin{vmatrix} A(4,0) & -1 & 0 & 0 & 0 \\ A(4,1) & n & -1 & 0 & 0 \\ A(4,2) & 0 & n & -1 & 0 \\ A(4,3) & 0 & 0 & n & -1 \\ A(4,4) & 0 & 0 & 0 & n \end{vmatrix} \qquad \text{(d)}$$

determinant of 5th order

Now generalization $4 \rightarrow n$.
Proof by mathematical induction.
http://en.wikipedia.org/wiki/Mathematical_induction
Comment:
The 2D determinant Pascal pattern representation (c)

$$\frac{\sum_{i=1}^{n} i^4}{\sum_{i=1}^{n} i^0 = n} = \frac{\begin{vmatrix} +n^0 & 1 & 0 & 0 & 0 \\ -n^1 & 1 & 2 & 0 & 0 \\ +n^2 & 1 & 3 & 3 & 0 \\ -n^3 & 1 & 4 & 6 & 4 \\ +n^4 & 1 & 5 & 10 & 10 \end{vmatrix}}{\begin{vmatrix} +1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 2 & 0 & 0 \\ +1 & 1 & 3 & 3 & 0 \\ -1 & 1 & 4 & 6 & 4 \\ +1 & 1 & 5 & 10 & 10 \end{vmatrix}} = 5!$$

Pascal's triangle determinants
of 5th order

is the **“king size” solution** of the given problem:
* beautiful, simple, visually convincing,
* **extensible, portable!**

"From a drop of water," said the writer, "a logician could infer the possibility of an Atlantic or a Niagara without having seen or heard of one or the other. So all life is a great chain, the nature of which is known whenever we are shown a single link of it. Like all other arts, the Science of Deduction and Analysis is one which can only be acquired by long and patient study nor is life long enough to allow any mortal to attain the highest possible perfection in it. Before turning to those moral and mental aspects of the matter which present the greatest difficulties, let the enquirer begin by mastering more elementary problems.”
Arthur Conan Doyle. A Study in Scarlet.
http://gutenberg.spiegel.de/doyle/studie/study.htm

Oh, more!

$$\frac{\sum_{i=1}^{n} i^6}{n} = \frac{\begin{vmatrix} (-n)^0 & 1 & 0 & 0 & 0 & 0 & 0 \\ (-n)^1 & 1 & 2 & 0 & 0 & 0 & 0 \\ (-n)^2 & 1 & 3 & 3 & 0 & 0 & 0 \\ (-n)^3 & 1 & 4 & 6 & 4 & 0 & 0 \\ (-n)^4 & 1 & 5 & 10 & 10 & 5 & 0 \\ (-n)^5 & 1 & 6 & 15 & 20 & 15 & 6 \\ (-n)^6 & 1 & 7 & 21 & 35 & 35 & 21 \end{vmatrix}}{\begin{vmatrix} (-1)^0 & 1 & 0 & 0 & 0 & 0 & 0 \\ (-1)^1 & 1 & 2 & 0 & 0 & 0 & 0 \\ (-1)^2 & 1 & 3 & 3 & 0 & 0 & 0 \\ (-1)^3 & 1 & 4 & 6 & 4 & 0 & 0 \\ (-1)^4 & 1 & 5 & 10 & 10 & 5 & 0 \\ (-1)^5 & 1 & 6 & 15 & 20 & 15 & 6 \\ (-1)^6 & 1 & 7 & 21 & 35 & 35 & 21 \end{vmatrix}}$$

(6+1) by (6+1) determinants

$$= p_6(n) = \frac{1}{42}(6n^6+21n^5+21n^4+11n^2+1)$$

Plot $p_6(x)$; $x \in \mathbb{R}$ and $p_6(z)$; $z \in \leq \mathbb{C}$.

$$\lim_{k\to\infty}\frac{\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ -1 & 1 & 3 & 3 & 0 & 0 & 0 \\ 0 & 1 & 4 & 6 & 4 & 0 & 0 \\ 1 & 1 & 5 & 10 & 10 & 5 & 0 \\ 0 & 1 & 6 & 15 & 20 & 15 & 6 \\ -1 & 1 & 7 & 21 & 35 & 35 & 21 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 3 & 0 & 0 & 0 \\ 0 & 1 & 4 & 6 & 4 & 0 & 0 \\ 0 & 1 & 5 & 10 & 10 & 5 & 0 \\ 0 & 1 & 6 & 15 & 20 & 15 & 6 \\ 0 & 1 & 7 & 21 & 35 & 35 & 21 \end{vmatrix}} =$$

$(2k+1=)$ by $(2k+1=7)$ determinants

42.612923374243529927... ???

$$\frac{\sum_{i=1}^{n}(-1)^{n-i}i^{6}}{1}=\frac{\begin{vmatrix} n \bmod 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ n^{1} & -1 & 2 & 0 & 0 & 0 & 0 \\ n^{2} & 1 & -2 & 2 & 0 & 0 & 0 \\ n^{3} & -1 & 3 & -3 & 2 & 0 & 0 \\ n^{4} & 1 & -4 & 6 & -4 & 2 & 0 \\ n^{5} & -1 & 5 & -10 & 10 & -5 & 2 \\ n^{6} & 1 & -6 & 15 & -20 & 15 & -6 \end{vmatrix}}{\begin{vmatrix} 1 \bmod 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1^{1} & -1 & 2 & 0 & 0 & 0 & 0 \\ 1^{2} & 1 & -2 & 2 & 0 & 0 & 0 \\ 1^{3} & -1 & 3 & -3 & 2 & 0 & 0 \\ 1^{4} & 1 & -4 & 6 & -4 & 2 & 0 \\ 1^{5} & -1 & 5 & -10 & 10 & -5 & 2 \\ 1^{6} & 1 & -6 & 15 & -20 & 15 & -6 \end{vmatrix}}$$

(6+1) by (6+1) determinants

$$D(k=7,n)=\begin{vmatrix} \rho^0=1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \rho^1 & 1 & 1 & 0 & 0 & 0 & 0 \\ \rho^2 & 0 & 3 & 1 & 0 & 0 & 0 \\ \rho^3 & 0 & 1 & 6 & 1 & 0 & 0 \\ \rho^4 & 0 & 0 & 5 & 10 & 1 & 0 \\ \rho^5 & 0 & 0 & 1 & 15 & 15 & 1 \\ \rho^6 & 0 & 0 & 0 & 7 & 35 & 21 \end{vmatrix}$$

7 by 7 determinant

$$\frac{\sum_{i=1}^{n}(-1)^{n-i}i^{2k}}{\sum_{i=1}^{n}(-1)^{n-i}i^{2}}=\frac{D(k,n)}{D(k,1)}\;;\;\rho=n(n+1)$$

$$\sum_{i=1}^{n}(-1)^{n-i}i^{2}=\sum_{i=1}^{n}i^{1}=\frac{1}{2}n(n+1)$$

$$\lim_{k\to\infty}\binom{2k+2}{2}\frac{D(k,0)}{D(k+1,0)}=\frac{\pi^2}{2}$$

$$B_{2k}=(-1)^{k-1}\frac{D(k,0)}{2(2^{2k}-1)}$$ Bernoulli numbers

$$D(k=7,n)=\begin{vmatrix} p^0=1 & 2 & 0 & 0 & 0 & 0 & 0 \\ p^1 & 1 & 3 & 0 & 0 & 0 & 0 \\ p^2 & 0 & 4 & 4 & 0 & 0 & 0 \\ p^3 & 0 & 1 & 10 & 5 & 0 & 0 \\ p^4 & 0 & 0 & 6 & 20 & 6 & 0 \\ p^5 & 0 & 0 & 1 & 21 & 35 & 7 \\ p^6 & 0 & 0 & 0 & 8 & 56 & 56 \end{vmatrix}$$

7 by 7 determinant

$$\frac{\sum_{i=1}^{n} i^{2k+1}}{\sum_{i=1}^{n} i^3} = \frac{D(k,n)}{D(k,1)} \; ; p = n(n+1)$$

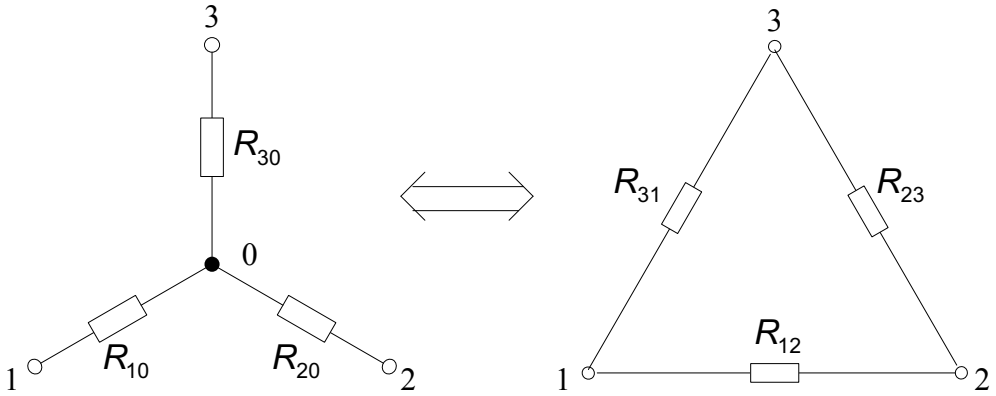
$$\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i^1\right)^2 = \frac{p^2}{4}$$

$$\lim_{k\to\infty} \binom{(2k+1)+3}{3} \frac{D(k,0)}{D(k+1,0)} = \frac{4}{3}\pi^2$$

Example 2
The bidirectional star-delta conversion, a standard problem of electrical system analysis

Example 1 showed: creative and efficient formula manipulation is an important part of mathematical description and analysis of signals. In the following we discuss from that point of view a standard problem of electrical system analysis.

Find a simplest way (minimal number of intermediate steps) to derive an as simple as possible equation system for the bidirectional star-delta conversion:



Star circuit with the Resistors R_{10}, R_{20} and R_{30}

Delta circuit with the Resistors R_{12}, R_{23} and R_{31}

Both circuits are equivalent regarding the terminals 1, 2, and 3. We assume the terminals are open, the circuits aren't part of a larger electrical network.

Solution

1. The equivalent resistance between the terminals 1 and 2 of the star circuit

$$Z_{star12} = R_{10} + R_{20} \qquad (1)$$

and of the delta circuit

$$Z_{delta12} = R_{12} \text{ parallel } (R_{23} + R_{31}) = \frac{R_{12}(R_{23} + R_{31})}{R_{12} + R_{23} + R_{31}} \qquad (2)$$

are equal. So we have

$$R_{10} + R_{20} = \frac{R_{12}R_{23} + R_{12}R_{31}}{R_{12} + R_{23} + R_{31}} \; . \qquad (3)$$

2. In the same way we obtain from

$$Z_{star23} = Z_{delta23} \text{ and } Z_{star31} = Z_{delta31} \; , \qquad (4)$$

or simpler by cyclic permutation in equ. (3) (the given problem is symmetrical)

$$R_{20} + R_{30} = \frac{R_{12}R_{23} + R_{23}R_{31}}{R_{12} + R_{23} + R_{31}} \; , \qquad (5)$$

and

$$R_{10} + R_{30} = \frac{R_{12}R_{31} + R_{23}R_{31}}{R_{12} + R_{23} + R_{31}} \; . \qquad (6)$$

3. Step 1 und 2 were very simple, the more difficult problem is now to separate R_{10}, R_{20}, and R_{30} on the left sides of equations that right side are expressions only in R_{12}, R_{23}, and R_{31}.
To find an efficient way to solve problems of this kind, it's often very useful to colorencode the relevant variables (one-to-one mapping), in our simple case R_{10}, R_{20}, and R_{30} in the equs (3), (5), and (6):

$$R_{10} + R_{20} = \frac{R_{12}R_{23} + R_{12}R_{31}}{R_{12} + R_{23} + R_{31}} \qquad (3)$$

$$R_{20} + R_{30} = \frac{R_{12}R_{23} + R_{23}R_{31}}{R_{12} + R_{23} + R_{31}} \qquad (5)$$

$$R_{10} + R_{30} = \frac{R_{12}R_{31} + R_{23}R_{31}}{R_{12} + R_{23} + R_{31}} \; . \qquad (6)$$

So, it is obvious that we can separate $\begin{matrix} R_{10} \\ R_{20} \\ R_{30} \end{matrix}$ by the operations

equ. (3) +equ. (6) – equ. (5) $$R_{10} = \frac{R_{12}R_{31}}{R_{12}+R_{23}+R_{31}}$$

equ. (3) +equ. (5) – equ. (6) , obtaining $$R_{20} = \frac{R_{12}R_{23}}{R_{12}+R_{23}+R_{31}}$$

equ. (5) +equ. (6) – equ. (3) $$R_{30} = \frac{R_{23}R_{31}}{R_{12}+R_{23}+R_{31}}\ .$$

(7, 8, 9)

More:
The practice to color a problem and the way to its solution on your "inner display" (from black-white to color display!), in your mind, this is a very useful method to improve your formula handling skill, it's a creativity tool.
References:

- http://www.answers.com/synesthesia
- http://www.answers.com/Daniel%20Tammet

"Daniel Paul Tammet (b. January 31, 1979; London)
Experiencing numbers as colors or sensations is a well-documented form of synesthesia, but the detail and specificity of Tammet's mental imagery of numbers is unique. In his mind, he says, each number up to 10,000 has its own unique shape and feel, that he can "see" results of calculations as landscapes, and that he can "sense" whether a number is prime or composite. He has described his visual image of 289 as particularly ugly, 333 as particularly attractive, and pi as beautiful. Tammet not only verbally describes these visions, but also creates artwork, particularly watercolor paintings, such as his painting of Pi."
D. Tammet. Born on a Blue Day: Inside the Extraordinary Mind of an Autistic Savant.
Free Press 2007

4. The equs (7), (8), and (9) describe the delta to star conversion: $R_{12}, R_{23}, R_{31} \rightarrow R_{10}, R_{20}, R_{30}$, but this is only the half of the solution. To find the other part, the star to delta conversion $R_{10}, R_{20}, R_{30} \rightarrow R_{12}, R_{23}, R_{31}$, we had to solve the equ. system (7), (8), and (9) for R_{12}, R_{23}, and R_{31}. Terrible! However, there's a surprisingly simple way. Look at equ. (7):

$$R_{10} = \frac{R_{12}R_{31}}{R_{12}+R_{23}+R_{31}} \qquad (7)$$

If we multiply both sides of this equ. by R_{23} than we obtain

$$R_{10}R_{23} = \frac{R_{12}R_{23}R_{31}}{R_{12}+R_{23}+R_{31}} \quad , \qquad (10)$$

on the right side a symmetrical expression in R_{12}, R_{23}, and R_{31} (extension of an equation with the goal of symmetrization) invariant to cyclic permutation, the invariant of the delta circuit

$$R^2_{delta} = \frac{R_{12}R_{23}R_{31}}{R_{12}+R_{23}+R_{31}} \qquad (11.1)$$

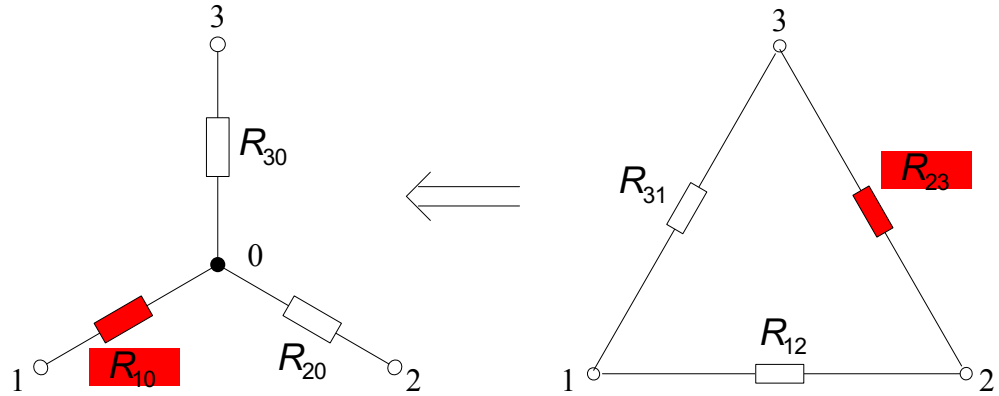
$$= \frac{1}{\frac{1}{R_{12}R_{23}}+\frac{1}{R_{12}R_{31}}+\frac{1}{R_{23}R_{31}}} \qquad (11.2)$$

$$\frac{1}{R^2_{delta}} = \frac{1}{R_{12}R_{23}}+\frac{1}{R_{12}R_{31}}+\frac{1}{R_{23}R_{31}} \qquad (11.3)$$

On the left side of equ. (10) we have the "opposite pair" $R_{10}R_{23}$

regarding the invariant R^2_{delta}:

$$R_{10}R_{23} = R^2_{delta} \qquad (12.1)$$



5. In the same way we obtain the formula system

$$R_{10}R_{23} = R_{delta}^2 \quad \rightarrow \quad R_{23} = \frac{R_{delta}^2}{R_{10}} \qquad (12.1,2)$$

$$R_{20}R_{31} = R_{delta}^2 \quad \rightarrow \quad R_{31} = \frac{R_{delta}^2}{R_{20}} \qquad (13.1,2)$$

$$R_{30}R_{12} = R_{delta}^2 \quad \rightarrow \quad R_{12} = \frac{R_{delta}^2}{R_{30}} \qquad (14.1,2))$$

The substitution of equs (12.2), (13.2), and (14.2) in equ. (11.2) results in

$$\frac{1}{R_{delta}^2} = \frac{1}{R_{12}R_{23}} + \frac{1}{R_{12}R_{31}} + \frac{1}{R_{23}R_{31}}$$

$$= \frac{1}{\frac{R_{delta}^2}{R_{30}}\frac{R_{delta}^2}{R_{10}}} + \frac{1}{\frac{R_{delta}^2}{R_{30}}\frac{R_{delta}^2}{R_{20}}} + \frac{1}{\frac{R_{delta}^2}{R_{10}}\frac{R_{delta}^2}{R_{20}}}$$

$$= \frac{1}{R_{delta}^4}(R_{10}R_{20} + R_{10}R_{30} + R_{20}R_{30})$$

and finally

$$R_{delta}^2 = R_{10}R_{20} + R_{10}R_{30} + R_{20}R_{30} = R_{star}^2 \quad !!! \qquad (15)$$

The values of the invariants R_{delta}^2 and R_{star}^2 are equal:

$$R_{delta}^2 = R_{star}^2 = R^2 \ . \qquad (16)$$

More:

To find and to handle invariants and opposite pairs (interdependence of opposites, unity in duality) in an efficient formula design process is a further useful creativity tool

References:

http://www.answers.com/topic/invariant-mathematics

http://www.answers.com/topic/invariant-physics

http://www.answers.com/Yin%20and%20Yang

By the way, the opposite-pair strategy is also a very effective approach to learn and to memorize vocabulary and sentences in a foreign language:

the day	the night
large	small

full empty
even number odd number
to transmit an encoded message to encode a received message
to switch on to switch off
the capacitor/capacitance the inductor/inductance
Great opportunity and great danger are two sides of the same coin.

6. Operation scheduling (step by step) in advance

A masterly chess-player plans always his moves in advance.

... the power of accurately calculating moves in advance has no greater place in chess than, perhaps, skilful calculation has in mathematics." .

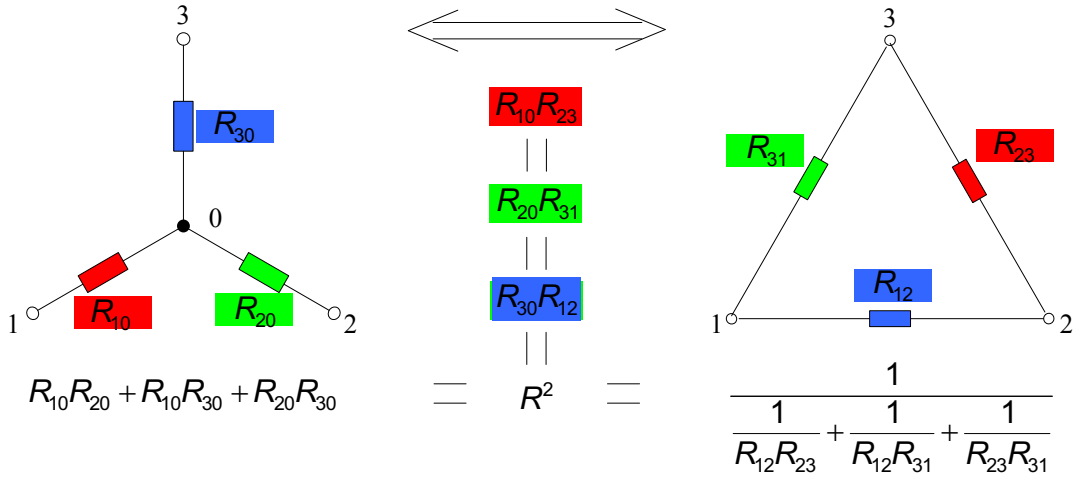
More:

http://www.chesscircle.net/forums/general-chess-forum/10897-how-many-moves-ahead.htm

http://seattlepi.nwsource.com/archives/1988/8801130570.asp

7. Result

Bidirectional star-delta conversion



In the treasure chest:

** Color encoding, inner color display, synesthesia
** Symmetrization
** Opposite pairs
** Invariants
** Operation scheduling in advance

Find more applications in mathematics, physics, electrical engineering, and computer science.

Example 5
Problemlösungstechniken

… ”nach geeigneter Umformung” “versuchsweise” “einen plausiblen Verteilungsansatz” machen ...

- Versuch! Kann, aber muß nicht das(ein) gewünschte(s) Ergebnis liefern!
- Immer versuchsbegleitend prüfen: löst das auf diesem Wege erhaltene Ergebnis wirklich das vorgelegte Problem? (theoretische bzw. experimentelle Verifikation)
- Wenn ja: erneuter Versuch: geht es noch einfacher?
- Wenn nein: neu konzipierter Versuch

Beispiel a)

- Zurückführen des Lösens der quadratischen Gleichung $x^2 + 2ax + b = 0$ auf das Lösen einer rein-quadratischen Gleichung $\rightarrow$
- Lösen der quadratischen Gleichung $x^2 + 2ax + b = 0$

$\boxed{x^2 + 2ax + b = 0}$; geeignete Umformung mittels Ansatz $\boxed{x = y + k}$ (zwei Unbekannte: y und k, aber nur eine Gleichung!)

$\rightarrow y^2 + 2yk + k^2 + 2ay + 2ak + b = y^2 + 2(k + a)y + (k^2 + 2ak + b) = 0$

Letztere quadratische Gleichung in y wird zur rein-quadratischen Gleichung in y (die sofort lösbar ist), wenn $k + a = 0$ gewählt wird $\rightarrow k = -a \rightarrow y = x + a$ (plausible Verteilung) $\rightarrow$

$y^2 + (a^2 - 2a^2 + b) = y^2 + (b - a) = 0 \rightarrow$

$$y^2 = a^2 - b = -\begin{vmatrix} 1 & a \\ a & b \end{vmatrix}$$

rein-quadratische Gleichung

$\rightarrow \; y_{1;2} = \pm\sqrt{a^2 - b} = x_{1;2} + a \; \rightarrow$

$$x_{1;2} = -a \pm \sqrt{a^2 - b}$$

Lösung der vorgelegten quadrat. Gleichung

Prüfung: Löst das erhaltene Ergebnis wirklich das vorgelegte Problem? $\rightarrow$
$x_{1;2} = 2ax_{1;2} + b = a^2 \mp 2a\sqrt{a^2 - b} + a^2 - b - 2a^2 \pm 2a\sqrt{a^2 - b} + b = 0$
$\rightarrow$ ja.

Beispiel b)

Zurückführen des Lösens der kubischen Gleichung $x^3 + 3ax^2 + 3bx + c = 0$ auf das Lösen einer rein-quadratischen Gleichung und einer rein-kubischen Gleichung

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$\boxed{x^3 + 3ax^2 + 3bx + c = 0}$; geeignete Umformung mittels Ansatz $\boxed{x = y + k}$ (zwei Unbekannte: y und k, aber nur eine Gleichung!)

$\rightarrow y^3 + 3y^2k + 3yk^2 + k^3 + 3ay^2 + 6ayk + 3ak^2 + 3by + 3bk + c =$

$$y^3 + 3(k + a)y^2 + 3(k^2 + 2ak + b)y + (k^3 + 3ak^2 + 3bk + c) = 0$$

Letztere kubische Gleichung in y wird um das quadratische Glied reduziert, wenn $k + a = 0$ gewählt wird $\rightarrow$

$k = -a \rightarrow y = x + a$ (plausible Verteilung) $\rightarrow$

$y^3 + 3(a^2 - 2a^2 + b)y + (-a^3 + 3a^3 - 3ab + c) = 0$

$$y^3 + 3\underbrace{(b - a\text{^}2\,)y}_{\begin{vmatrix} 1 & a \\ a & b \end{vmatrix} = p} + \underbrace{(2a^3 - 3ab + c)}_{-\begin{vmatrix} 1 & 2a & b \\ a & 2b & c \\ 0 & 1 & a \end{vmatrix} = q} = \boxed{\begin{array}{c} y^3 + 3py + q = 0 \\ \text{reduzierte kubische Gleichung} \end{array}}$$

geeignete Umformung mittels Ansatz $\boxed{y = u + v}$ (zwei Unbekannte: u und v, aber nur eine Gleichung!)

$\rightarrow u^3 + 3u^2v + 3uv^2 + v^3 + 3p(u + v) + q = 0$

$u^3 + v^3 + 3(uv + p)(u + v) + q = 0$

Letztere kubische Gleichung in u und v wird wesentlich reduziert, wenn $uv + p = 0$ gewählt wird $\rightarrow uv = -p$ (plausible Verteilung von u und v) $\rightarrow$

$(u^3 + v^3 = -q,\ uv = -p)\ \rightarrow$

$$(u^3 + v^3)^2 - 4u^3v^3 = \boxed{(u^3 - v^3)^2 = 4p^3 + q^2 = -\begin{vmatrix} 1 & 2a & b & 0 \\ a & 2b & c & 0 \\ 0 & 1 & 2a & b \\ 0 & a & 2b & c \end{vmatrix} \quad rein - quadratische\ Gleichung}$$

$\rightarrow (u^3 - v^3)_{1;2} = \pm\sqrt{4p^3 + q^2}\ \rightarrow$

$u^3 + v^3 = -q, (u^3 - v^3)_{1;2} = \pm\sqrt{4p^3 + q^2} \quad \overset{y = u + v!}{\rightarrow}$

$$\boxed{u^3 = \frac{1}{2}\left(-q + \sqrt{4p^3 + q^2}\right),\ \ v^3 = \frac{1}{2}\left(-q - \sqrt{4p^3 + q^2}\right) \quad rein - kubische\ Gleichung}$$

Aufgabe b)

Führen Sie den Lösungsprozeß nun bis zu Ende durch und geben Sie geschlossene Lösungsformeln für $x_{1;2;3}$ unter Benutzung von Wurzeln und Determinanten an. Diskutieren Sie Spezialfälle.

Lösung:

$x_1 = u + v;\ \left.\begin{matrix} u \\ v \end{matrix}\right\} = \sqrt[3]{-\frac{q}{2} \pm \sqrt{p^3 + \left(\frac{q}{2}\right)^2}}$ (Cardanische Formel [90,91,92])

$$= \sqrt[3]{\frac{1}{2}\sqrt{D_3 \pm \sqrt{-D_4}}}\ ;$$

$$D_2 = \begin{vmatrix} 1 & a \\ a & b \end{vmatrix};\ D_3 = \begin{vmatrix} 1 & 2a & b \\ a & 2b & c \\ 0 & 1 & a \end{vmatrix};\ D_4 = \begin{vmatrix} 1 & 2a & b & 0 \\ a & 2b & c & 0 \\ 0 & 1 & 2a & b \\ 0 & a & 2b & c \end{vmatrix}$$

$$x_2 = \varepsilon_2 u + \varepsilon_3 v;\ \ x_3 = \varepsilon_3 u + \varepsilon_2 v;\ \ \varepsilon_{2;3} = \frac{1}{2}(-1 \pm j\sqrt{3})$$

D_2	D_2	D_2	Klassifikation der Lösungen(Wurzeln) von $x^3 + 3ax^2 + 3bx + c = 0$
		> 0	3 voneinander verschiedene reelle Wurzeln (casus irreducibilis!)
		< 0	1 reelle Wurzel, 1Paar konjugiert komplexe Wurzeln
$\neq$		$= 0$	3 reelle Wurzeln, davon zwei gleiche: $x_1 = -a - \frac{D_3}{D_2};\ x_{2;3} = -a - \frac{D_3}{2D_2}$
$= 0$	$= 0$	$= 0$	3 reelle Wurzeln, davon zwei gleiche: $x_{1;2;3} = -a$